Lecture 2

σ-algebras.

$\{E \subseteq X\}$ → $\mathcal{M} \subseteq \mathcal{P}(X)$ ← some fixed set

Def 1. A collection of subsets $\mathcal{M} \subseteq \mathcal{P}(X)$ is a *σ-algebra* if

(i) $E_k \in \mathcal{M}$, $k = 1, 2, \ldots \Rightarrow \bigcup_{k=1}^{\infty} E_k \in \mathcal{M}$

(ii) $E \in \mathcal{M} \Rightarrow E^c := X \setminus E \in \mathcal{M}$.

Rem 1. Since $\bigcap_{k=1}^{\infty} E_k = \left(\bigcup_{k=1}^{\infty} E_k^c \right)^c$, a σ-algebra is also closed under countable intersections.

The collection $\mathcal{A} \subseteq \mathcal{P}(X)$ is an *algebra* if it is closed under *finite* intersections and taking complements.

If $\mathcal{E} \subseteq \mathcal{P}(X)$ is any collection of subsets, then $\mathcal{M}(\mathcal{E})$ is the smallest σ-algebra that contains $\mathcal{E}$ (this is well defined as intersections of σ-algebras is easily seen to be a σ-algebra).

$\mathcal{M}(\mathcal{E})$ is called the <u>σ-algebra generated by $\mathcal{E}$</u>.

An obvious but useful fact:
If $\mathcal{E} \subseteq \mathcal{M}(\mathcal{F})$, then $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{F})$.

<u>Borel σ-algebra</u>.

If X is a topological space (e.g. a metric space), then the σ-algebra generated by the open sets is called the <u>Borel σ-algebra</u> and denoted $\mathcal{B}_X$.

<u>Prop 1</u>. $\mathcal{B}_{\mathbb{R}}$ is generated by the open or the closed or the half-open intervals.

<u>Pf</u>. DIY.

If we go back to Prop. 1, we note that we might hope to define a measure on $\mathcal{B}_{\mathbb{R}}$ that satisfies (i)–(iii). This turns out to be possible (even with (iii) replaced by (iii') $\mu([a,b)) = b-a$). This is (essentially) the Lebesgue measure. (There is another technicality that we will get to later.)

Product σ-algebras.

some index set

Recall that if X_α, $\alpha \in A$, are given sets, the Cartesian product $X = \prod_{\alpha \in A} X_\alpha$ is defined as the set of functions f on A s.t. $f(\alpha) \in X_\alpha$.

X is non-empty by Axiom of Choice.

We also have projections $\pi_\alpha : X \to X_\alpha$ taking $f \to f(\alpha)$.

If $\mathcal{M}_\alpha$ are σ-algebras on X_α, then the <u>product σ-algebra</u>

$\mathcal{M}_X = \bigotimes_{\alpha \in A} \mathcal{M}_\alpha$ on X is generated by

$$\mathcal{E}_X = \{ \pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{M}_\alpha, \alpha \in A \}$$

<u>Prop 2.</u> If A is countable (or finite), then $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$ is generated by

$$\mathcal{E}_X' = \{ \prod_{\alpha \in A} E_\alpha : E_\alpha \in \mathcal{M}_\alpha, \alpha \in A \}$$

<u>Pf.</u> Clearly, $\mathcal{E}_X \subseteq \mathcal{E}_X' \Rightarrow \mathcal{M}'(\mathcal{E}_X) \subseteq \mathcal{M}(\mathcal{E}_X) \subseteq \mathcal{M}(\mathcal{E}_X')$.

For reverse inclusion, note that for any collection $E_\alpha \in \mathcal{M}_\alpha$, we have $\prod_{\alpha \in A} E_\alpha = \bigcap_{\alpha \in A} \pi_\alpha^{-1}(E_\alpha)$.

Thus, if A is <u>countable</u>, by Rem 1 above, $\prod_{\alpha \in A} E_\alpha \in \mathcal{M}(\mathcal{E}_X) \Rightarrow$

$\mathcal{M}(\mathcal{E}_X') \subseteq \mathcal{M}(\mathcal{E}_X)$. ▨

<u>Rem 2:</u>
Similarly, one can show that if $\mathcal{M}_\alpha$ is generated $\mathcal{E}_\alpha$ then $\bigotimes_A \mathcal{M}_\alpha$ is generated by $\pi_\alpha^{-1}(E_\alpha)$, $E_\alpha \in \mathcal{E}_\alpha$, and if A countable by $\prod_A E_\alpha$, $E_\alpha \in \mathcal{E}_\alpha$.

Borel σ-algebra, revisited.

<u>Prop 3.</u> If $X_1, \dots, X_n$ are metric (top.) spaces and $X = X_1 \times \dots \times X_n$, then $\bigotimes_{k=1}^{n} \mathcal{B}_{X_k} \subseteq \mathcal{B}_X$. If X_k are separable, then $\bigotimes_{k=1}^{n} \mathcal{B}_{X_k} = \mathcal{B}_X$.

<u>Pf.</u> DIY. ⊠

<u>Cor 1.</u> $\mathcal{B}_{\mathbb{R}^n} = \bigotimes_{k=1}^{n} \mathcal{B}_{\mathbb{R}}$.

Elementary families.

<u>Def 2.</u> An <u>elementary family</u> $\mathcal{E}$ is a collection of subsets $E \subseteq X$ s.t.

(i) $\emptyset \in \mathcal{E}$

(ii) $E, F \in \mathcal{E} \Rightarrow E \cap F \in \mathcal{E}$

(iii) $E \in \mathcal{E} \Rightarrow E^c := X \setminus E$ is a finite union of sets in $\mathcal{E}$.

Prop 4. If $\mathcal{E}$ is an elementary family, then the collection of finite disjoint unions of set in $\mathcal{E}$ forms an algebra $\mathcal{A}$.

Pf. DIY. $\boxed{\mathcal{E}}$

Measures.

A set X w/ σ-algebra $\mathcal{M}$ is called a measurable space.

A measure on $(X, \mathcal{M})$ is a function $\mu : \mathcal{M} \to [0, \infty]$ s.t.

(i) $\mu(\emptyset) = 0$

(ii) $\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$ when

$$E_k \cap E_\ell = \emptyset \text{ for } k \neq \ell.$$